EFFECT OF VIBRATIONS ON THE MOTION OF SMALL GAS BUBBLES IN A LIQUID

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SUMMARY

An understanding has been obtained of the mechanism governing the motion (migration) of small gas bubbles in vibrated vessels described in the Introduction. Differential equations describing the behavior were derived, equations (27) and (28), under the restrictive assumption of an inviscid fluid; from qualitative considerations it was concluded that viscosity does not affect the mechanism of migration, but may have quantitative effects. A criterion when the influence of viscosity will be appreciable is given. The situation is discussed at the end of Section I when the results of the analysis are compared with the pilot tests.

No attempt has been made to solve the nonlinear differential equations in general. Instead, locations in space were determined in which the differential equations have time periodic solutions, representing small oscillations of the bubbles. The loci of these solutions form surfaces which separate the tank into regions of different bubble behavior. In case of a rigid tank it was found that the bubbles above a certain level h will rise towards the surface, while those below will sink to the bottom. In the case of elastic vessels the regions have complicated shapes, discussed for cylindrical tanks in Section II and shown in figures 6 and 7. From the character of the regions it is concluded that, if many bubbles are present, clusters of bubbles will collect in certain locations near the wall or bottom.

This report has been concerned solely with the behavior of a single small bubble. It will be followed by a second report describing a generalization of the analysis applicable to clusters of bubbles, such as have been observed in the tests reported in Appendix A. The second report will also include a more detailed study of the effect of surfaces and of the finite size of the vessel on clusters of bubbles.

The analysis presented here indicates that the region just below the surface should always be one in which bubbles rise towards the surface and vent. The theory therefore does not yet contain the explanation for the streams of small bubbles moving from the surface towards the bottom which were observed at high accelerations (see Appendix A). It is suspected by the writer that this phenomenon may be caused by the pressure field of the sloshing motion which was present whenever the bubble stream was seen. Further work in this direction is required.

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INTRODUCTION

If a transparent vessel, partially filled with water, is vibrated in the vertical direction complicated phenomena can be observed. If the vibration is sufficiently strong, small individual gas bubbles created by surface disturbances appear in the lower part of the vessel.* These bubbles do not rise to the surface but move along the bottom or side, or perform vibratory motions in the middle of the vessel. It is the purpose of this paper to contribute to the understanding of the complex phenomena** observed by presenting an analysis for the motion of a gas bubble in a vibrated vessel.

To demonstrate the fact that gas bubbles in a vibrated tank may not rise to the surface as one would expect from considerations of buoyancy, the following simple experiment was made. A rubber skinned test balloon of about 1/4-inch diameter was attached to a short piece of string at the end of a wire; this balloon was inserted in a cylindrical transparent plastic test tank, figure 1, placing it on the centerline of the tank about 2 inches above the bottom. Without vibration the bubble rises, stretching the string vertically. Vibrating the tank vertically with gradually increasing acceleration, the position of the bubble on the centerline became unstable and snapped into a deflection position I, see figure 2. A slight further increase of the acceleration made this position also unstable, and the bubble went into position II at the bottom of the tank. If the acceleration was decreased by about 1 g the bubble returned to its original position. The acceleration required to make the bubble sink was somewhat frequency dependent, as may be seen from Table 1 giving the acceleration required to hold the bubble in position II, figure 2.

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* This fact was first observed by Y. C. Lee, Principal Engineer, and C. C. Miesse, Physicist, Aerojet-General Corporation, who brought the matter to the writer's attention.

** A short description is attached as Appendix A.

TABLE 1

Cycles/second	60	90	120	150	175	200
Acceleration	16g	14.5g	14g	13g	12g	11.5g

The somewhat unexpected sinking of bubbles--contrary to buoyancy--has its counterpart in the field of underwater explosions. It is known from extensive work on this subject that pulsating explosion bubbles do not simply rise, but move in a modified manner known as "bubble migration." Due to migration an explosion bubble may in certain cases sink, instead of rise. It can therefore be expected that an analysis using methods similar to those employed in the explosion field will give an understanding of the phenomena observed.

In the case under consideration the vibration will excite pulsations (meaning changes in radius) of the bubble and it will be shown that migration effects do occur which modify the path of the bubble and may even cause it to sink. Specific attention will be given to the case of a bubble moving vertically in an oscillatory manner, a situation which separates the cases of rising and sinking bubbles.

The migration phenomena of explosion bubbles can be explained by an analysis using an incompressible, inviscid fluid, Herring[1] and Taylor,[2] and Bryant,[3] except that such an analysis does not explain the decay of the bubble pulsations due to radiation. As this decay affects the migration of the explosion bubble only in a secondary manner it is usually considered as a correction. In the present problem it is intended to obtain the response of the bubble for cases where the amplitude of the pulsation is much smaller than in the explosion case, and radiation effects need therefore not be included at all. It is also assumed that the frequency of the forced oscillations of the vessel is small versus the natural frequency of the bubble; this excludes the possibility of resonance of the bubble pulsations, which would have required compressibility as a damping mechanism. The present analysis, similar to the references quoted, does not consider changes in the spherical shape of the bubble. The assumption of spherical bubbles is confirmed by observation on explosion bubbles, except in their most contracted stage. As only relatively small pulsations will occur in the present case any deviations from spherical shape can be expected to be minor and unimportant. This reasoning is supported by the fact that no visibly non-spherical bubbles were observed in the tests.

Migration is a second order effect which cannot be explained by a linear theory of small vibrations. It is therefore necessary to obtain equations of motion for large displacements which, even for an approximate analysis, cannot be completely linearized.

To reduce the problem to its simplest form, the behavior of a bubble in a large rigid vessel is considered first, Section I. It is assumed that the bubble is sufficiently far away from any surface--at least several bubble diameters--such that interactions may be neglected. The results obtained are generalized in Section II to allow for the quite important effect of the elasticity of the vessel, and--qualitatively--for the effect of proximity of surfaces and of viscosity.

SECTION I

BEHAVIOR OF A GAS BUBBLE IN A LARGE RIGID VESSEL

It is intended to study the motion of a small spherical gas bubble in an incompressible, inviscid fluid if the vessel containing the fluid is vibrated in the vertical direction. It is assumed, in this section, that the vessel is rigid and that a constant ullage pressure p_o will be maintained above the surface of the fluid; it is further assumed that the bubble is sufficiently far away from any surface--at least several diameters--such that interactions are negligible.

Equations of Motion

Excluding rotational motions of the fluid the state of the system shown in figure 3 is fully described by three generalized coordinates:

$x(t)$ - vertical position of vessel with respect to a reference line
$\Delta(t)$ - increase of bubble radius above a reference radius a
$z(t)$ - depth of bubble below surface

To obtain Lagrange's equations, expressions for the kinetic and potential energies of the system are required. The bubble being very small compared to the dimensions of the vessel, and far from any surface, the kinetic energy is computed by utilizing the known virtual mass expressions for a sphere in an infinite fluid in the following manner. The total velocity at any point is

$$\vec{v} = \vec{v}_x + \vec{v}_\Delta + \vec{v}_z$$

where $\vec{v}_x = \dot{x}$ is the vertical velocity of the vessel which is not a function of the location. $\vec{v}_\Delta$ and $\vec{v}_z$ are the velocities due to the coordinates Δ and z. Let dV be the element of volume, ρ the density of the fluid; the kinetic energy T is then

$$T = \frac{\rho}{2}\int \vec{v}^2 dV = \frac{\rho}{2}\dot{x}^2\int dV + \frac{\rho}{2}\int \vec{v}_\Delta^2 dV + \frac{\rho}{2}\int \vec{v}_z^2 dV + \rho\dot{x}\int (u_\Delta + u_z)\, dV \quad (1)$$

where u_Δ and u_z are the vertical components of $\vec{v}_\Delta$ and $\vec{v}_z$, respectively.

There is no coupling term between $\vec{v}_\Delta$ and $\vec{v}_z$ because of symmetry. The value of the first term in Eq. (1) is simply $M\dot{x}^2/2$, where M is the total mass of the fluid in the vessel; the second and third terms are the familiar expressions for the kinetic energies of the respective motions of a sphere of radius $a + \Delta$ (reference 4, pp. 122, 124); while the last term, a coupling term, is evaluated in Appendix B. The total kinetic energy is, therefore,

$$T = \frac{M}{2}\dot{x}^2 + 2\pi\rho(z+\Delta)^3\dot{\Delta}^2 + \frac{\pi}{3}\rho(a+\Delta)^3\dot{x}^2 - \frac{4\pi}{3}\rho\dot{x}\frac{\partial}{\partial t}\left[(a+\Delta)^3 z\right] \quad (2)$$

The potential energy consists of three parts: the potential of the gravity field, $-gMx + g\rho\frac{4\pi}{3}(a+\Delta)^3 z$; the potential of the gas above the surface, $\frac{4\pi}{3}(a+\Delta)^3 p_o$; the potential of the gas inside the bubble, which can be computed from the pressure-volume relation

$$p(a+\Delta)^{3\gamma} = p_1 a^{3\gamma} \quad (3)$$

where γ is the ratio of the specific heats, and p_1 the pressure in the bubble when its size equals the reference radius a. The potential becomes

$$\int_v^\infty p\,dv = \frac{4\pi}{3}\,\frac{p_1}{\gamma-1}\,\frac{p_1 a^{3\gamma}}{(a+\Delta)^{3\gamma-3}} \quad (4)$$

Collecting all terms:

$$P = -gMx + \frac{4\pi}{3}(a+\Delta)^3(p_o + g\rho z) + \frac{4\pi}{3(\gamma-1)}\,\frac{p_1 a^{3\gamma}}{(a+\Delta)^{3\gamma-3}} \quad (5)$$

The general form of Lagrange's equation is

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = Q_n \quad (6)$$

where $L = T - P$; the coordinates q_n are in turn Δ, z and x, while

Q_n are the respective generalized forces. In the present case all forces have been included in the potential P except the external driving force producing the oscillation. As this force does no work if Δ and z change, we have $Q_\Delta = Q_z = 0$, while Q_x does not vanish. If one visualizes a situation where the displacement x is prescribed, Eq. (6), with respect to the coordinate, x is not required because it serves only to determine the force Q_x; this leaves two differential equations for the two unknown functions Δ and z:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\Delta}} - \frac{\partial L}{\partial \Delta} = 0 \quad ; \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

After substitution,

$$\frac{\partial}{\partial t}\left[(a+\Delta)^3 \dot{z}\right] = 2(a+\Delta)^3(\ddot{x} - g) \tag{7}$$

$$\ddot{\Delta} + \frac{3\dot{\Delta}^2}{2(a+\Delta)} - \frac{p_1 a^{3\gamma}}{\rho(a+\Delta)^{3\gamma+1}} + \frac{(g-\ddot{x})z}{a+\Delta} - \frac{\dot{z}^2}{4(a+\Delta)} = -\frac{p_o}{\rho(a+\Delta)} \tag{8}$$

Discussion of Equation (7)

This equation represents the principle of conservation of momentum, and it is useful to consider it for the simple case when the vessel is at rest, $\ddot{x} = 0$. Using the bubble volume $v = \frac{4\pi}{3}(a+\Delta)^3$ as variable, Eq. (7) may be written

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2} v \dot{z}\right) = -\rho g v$$

where the term $\rho/2\, v$ is the virtual mass of the bubble (for the z-motion), while $-\rho g v$ is the buoyancy. If v changes with time, the velocity $\dot{z}$ is

$$\dot{z} = -2g\frac{1}{v}\int_{t_o}^{t} v\, dt$$

Due to the fact that the bubble volume v is necessarily positive, the integral will always increase with t; the term $1/v$ will increase the velocity $\dot{z}$ if v is smaller, or decrease it if v is larger than average, but $\dot{z}$ will always remain negative and the bubble will rise continuously.

Now consider the situation if a time-dependent acceleration $\ddot{x}(t) = Ng \cos \omega t$ is imposed where N is a number. Equation (7) becomes

$$\dot{z} = -2g \frac{1}{v} \int_{t_o}^{t} \left(v - vN \cos \omega t\right) dt \tag{9}$$

It is immediately apparent that the integral will not continuously increase if $N > 1$. To show that the integral can even become negative; that is, that the bubble can sink, let the bubble execute an imposed periodic pulsation $v = v_o(1 + \alpha \cos \omega t)$ where α is a number, defining the magnitude of the pulsations. By substitution

$$\int_{t_o}^{t} (v - vN \cos \omega t)\, dt = v_o \int_{t_o}^{t} \left[1 - \frac{\alpha N}{2} + N \cos \omega t - \frac{\alpha N}{2} \cos 2\omega t\right] dt \tag{10}$$

The terms $\cos \omega t$ and $\cos 2\omega t$ under the integral give oscillatory contributions, while the first two terms give a monotone contribution which describes the direction of the over-all motion. If $\alpha N > 2$, the integral will ultimately be negative and the bubble will sink; if $N\alpha = 2$, the bubble will execute an oscillation about a mean position.

The above consideration shows the possibility of bubbles sinking, provided the bubble volume changes in the prescribed manner. It remains to be shown that the imposed vibration will produce the assumed or an equivalent pulsation. This requires simultaneous consideration of the two Lagrangian equations (7) and (8).

Oscillation of a Bubble Around a Mean Position

Restricting the general problem of the motion of the bubble, it is asked whether there are combinations of imposed acceleration $\ddot{x} = Ng \cos \omega t$ and bubble size and location for which a bubble will undergo a periodic vertical motion and pulsation. The previous discussion has shown that for large N (say $N \sim 10$) only small volume changes $\alpha = 2/N$ are required; the equations can therefore be linearized with respect to the change of radius Δ.

It is convenient to replace the coordinate z by $z = h + \xi$, where h is the depth at which a bubble of the reference radius a would be in equilibrium if prevented from rising; this gives the relation $p_1 = p_o + hg\rho$. Equations (7) and (8) can then be arranged as follows:

$$\frac{\partial}{\partial t}\left[(a+\Delta)^3 \dot{\xi}\right] = 2(a+\Delta)^3(\ddot{x} - g) \tag{11}$$

$$\ddot{\Delta} + \frac{3}{2}\frac{\dot{\Delta}^2}{a+\Delta} + \frac{p_1}{\rho(a+\Delta)}\left[1 - \left(\frac{a}{a+\Delta}\right)^{3\gamma}\right] + \frac{1}{a+\Delta}(g - \ddot{x})\xi - \frac{\dot{\xi}^2}{4(a+\Delta)} = \frac{h}{a+\Delta}\ddot{x} \tag{12}$$

If Δ is small, the second term of Eq. (12) can be dropped and the third term becomes, by expansion in powers of Δ

$$\frac{3\gamma p_1}{a^2\rho}\Delta \equiv \Omega^2\Delta \tag{13}$$

where Ω is the frequency of small oscillation of a bubble of radius a at the pressure p_1 (reference 1, page 79). If the terms containing ξ and $\dot{\xi}^2$ in Eq. (12) could also be neglected, which will be seen to be permissible later, Eqs. (11) and (12) become

$$\frac{\partial}{\partial t}\left[(a+3\Delta)\dot{\xi}\right] = 2(a+3\Delta)(\ddot{x} - g) \tag{14}$$

$$\ddot{\Delta} + \Omega^2\Delta = \frac{h}{a}\ddot{x} \tag{15}$$

If $\ddot{x} = Ng \cos \omega t$, the second equation has a solution of the form

$$\Delta = \frac{\alpha}{3} a \cos \omega t \tag{16}$$

where

$$\alpha = \frac{3h}{a^2} Ng \frac{1}{\Omega^2 - \omega^2} = \frac{h\rho}{\gamma p_1} Ng \frac{1}{1 - \frac{\omega^2}{\Omega^2}} \tag{17}$$

Substitution of Eq. (16) into Eq. (14) gives

$$\begin{aligned} \frac{\partial}{\partial t}\left[(1 + \quad \cos \omega t)\dot{\xi}\right] &= 2(1 + \alpha \cos \omega t)(Ng \cos \omega t - g) \\ &= (\alpha N - 2)g + 2Ng \cos \omega t + \alpha Ng \cos 2\omega t - 2\alpha g \cos \omega t \end{aligned} \tag{18}$$

which leads to an oscillatory solution only if

$$\alpha N = 2$$

In this case

$$\dot{\xi} = \frac{Ng}{\omega}\left(2 \sin \omega t - \frac{3\alpha}{2} \sin 2\omega t \ldots\right) \tag{19}$$

where terms containing higher power of α were omitted. Further,

$$\xi = -\frac{Ng}{\omega^2}\left(2 \cos \omega t - \frac{3\alpha}{4} \cos 2\omega t \ldots\right) \tag{20}$$

The oscillatory solution visualized can only have physical meaning if the amplitude ξ of the vertical motion remains smaller than the depth h, as the bubble would otherwise vent; this requires

$$\omega^2 \gg \frac{Ng}{h} \tag{21}$$

To justify the dropping of the terms containing ξ when obtaining Eq. (15), the neglected terms can be estimated for $N \gg 1 \gg \alpha$:

$$\frac{1}{a+\Delta}\left[(g-\ddot{x})\xi - \frac{1}{4}\dot{\xi}^2\right] \sim \frac{N^2 g^2}{4 a \omega^2}(1 + 3\cos 2\omega t) = 0\left(\frac{N^2 g^2}{a\omega^2}\right) \tag{22}$$

This can be compared with the right-hand side of Eq. (12), which is of the order

$$\frac{h}{a+\Delta}\ddot{x} = 0\left(\frac{hNg}{a}\right) \tag{23}$$

It is easily seen that the necessary condition (21) automatically ensures that the expression (22) is small versus (23). The neglected terms can therefore never be of consequence for the oscillatory solutions contemplated.

Equations (17) and (18) can be solved for the value h, noting $p_1 = p_o + \rho g h$:

$$h = \frac{p_o}{\rho g} \frac{1}{\dfrac{N^2 \Omega^2}{2\gamma(\Omega^2 - \omega^2)} - 1} \tag{24}$$

the solution being meaningful only if Eq. (21) is satisfied. The size of the bubble appears only in the value of the frequency Ω, and because for small bubbles usually $\Omega^2 \gg \omega^2$, Eq. (24) simplifies to

$$h = \frac{p_o}{\rho g} \frac{2\gamma}{N^2 - 2\gamma} \tag{25}$$

Equation (25) furnishes positive values h for any value $N > \sqrt{2\gamma}$; but because $\alpha = 2/N$ was assumed to be small, h may be inaccurate unless N is much larger than $\sqrt{2\gamma}$. Equation (21) then defines frequencies ω^2 above which the depth found will apply. If ω is comparable to Ω, Eq. (24) must be used; however the form of this equation is such that h becomes negative if $\omega > \Omega$, and no oscillatory solution exists if $\omega > \Omega$; as a rule ω must be even noticeably smaller than Ω, otherwise Eq. (21) will not be satisfied.

Stability of the Oscillatory Solution

It is important to realize that the oscillatory solution just obtained is unstable; that is, one cannot expect to observe bubbles executing oscillations on the level h given by Eqs. (24) and (25). To prove the instability consider Eq. (17). If h is slightly smaller than the required critical value, α will also be smaller than required to satisfy the condition $\alpha N = 2$; the first term of Eq. (18) will then be negative such that the bubble has an average upward velocity; that is, it will move away from the level h of steady oscillations. If h were slightly larger, the bubble would similarly have an average downward velocity, again away from the level h. In case of any disturbance, regardless in which direction, the bubble will not return to its original motion; i.e., the motion is unstable.

In spite of the fact that the motion described by the solution found has therefore no physical reality, the critical level h for the unstable solution has an important meaning: bubbles above the critical level h will rise to the surface, while those below will sink. This conclusion is in qualitative agreement with observations.

It might already be stated at this point that it will be seen in Section II that the assumption of a rigid vessel on which the above discussion is based is an oversimplification, and that a more refined analysis may furnish more than one level of oscillatory solutions, some of which are stable.

Comparison with the Pilot Tests

The result obtained, Eqs. (24) and (25), can be compared with the result of the simple test reported in Table 1. To obtain a fair comparison the effect of the weight of the rubber skin should be allowed for, and it can be seen easily that the added weight will reduce the critical depth h.* One should expect, therefore, that the values h computed from Eq. (24) for the appropriate values N of the acceleration should be larger than the test depth of the bubble, $h \simeq 7$ inches. The natural frequency Ω of the 1/4-inch bubble tested is so much higher ($\Omega > 1000$ cycles/sec) than the test frequencies ω, that Eq. (25) may be used; the computed values of h are shown in Table 2, using $p_o/\rho g = 33$ feet and $\gamma = 1.4$.

TABLE 2

(h in inches, $\dot{\xi}$ in inches/sec)

Cycles per second	60	90	120	150	175	200
From Table 1: N =	16	14.5	14	13	12	11.5
From Eq. (24): h =	4.4	5.4	5.8	6.7	7.9	8.7
$\dot{\xi} \simeq 2Ng/\omega$ =	32	20	14	11	8.5	7

While the computed values h are of the expected order of magnitude, not all are larger than h = 7 in., as predicted by the theory. It appears that the lack of agreement can be ascribed to the unrealistic assumption of an inviscid fluid, which is not justified for the entire range of the test. To show this, the last line of Table 2 contains the maximum velocities $\dot{\xi}$ of the bubble, computed from the first term of Eq. (19), $\dot{\xi} \simeq 2Ng/\omega$. The problem of viscosity effects is discussed in Appendix C, where it is

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* h will be reduced to $h(1 - \beta)$ where β is the ratio of the weight of the skin to the buoyancy of the bubble. The ratio β for the test case is not known, but was presumably about 0.1 - 0.2.

concl· ded that such effects are small provided the velocity of the bubble remains small compared to its terminal velocity when rising in a gravity field; for the present case this velocity is of the order of 10 in./sec, and the inviscid analysis cannot be expected to give a good value for the depth h, except for high frequencies, $\omega > 150$ cycles/sec. It is also concluded in Appendix C that viscosity will increase the value of the critical depth h; in the limiting case, the actual depth would be three times the value according to Eqs. (24) and (25). The actual depth, $h = 7$ inches, being larger than the computed ones shown in Table 2, the direction of the differences is in accordance with this prediction.

SECTION II
MOTION OF A GAS BUBBLE IN AN ELASTIC VESSEL

Equations of Motion

Maintaining all other assumptions made in the previous section, the effect of the elasticity of the vessel on a small bubble can only originate from the modified pressure field in the elastic vessel. In the rigid vessel the pressure at any instant is a linear function of the depth below the surface only, while in an elastic vessel the pressure is a nonlinear function of all three coordinates.

It is not necessary to treat the problem of the vessel and the bubble as a unit. Let the pressure field in the vibrated elastic vessel without the bubble be known; because of the fact that the motion of a small bubble can essentially depend only on the pressure field in its immediate vicinity, one can use the equations of motion derived for the case of a rigid vessel for any case, provided one identifies the significant factors; that is, the pressure $\bar{p}$ due to the vibration and its gradient. For the present purpose it is sufficient to use the simplified Eqs. (14) and (15). The pressure $\bar{p}$ due to the vibration for the case of a rigid vessel is

$$\bar{p} = -\rho h \ddot{x}, \qquad \mathrm{grad}\,\bar{p} = \frac{\partial \bar{p}}{\partial h} = -\rho \ddot{x} \tag{26}$$

and the equations of motion can be rewritten

$$\Delta + \Omega^2 \Delta = -\frac{\bar{p}}{a\rho} \tag{27}$$

$$\frac{\partial}{\partial t}\left[(a + 3\Delta)\dot{\xi}\right] = -2(a + 3\Delta)\left(\frac{1}{\rho}\frac{\partial \bar{p}}{\partial h} + g\right) \tag{28}$$

where $\dot{\xi}$ is the vertical component of the bubble velocity. Similar equations for the horizontal velocities can be written, but do not contain the term g.

Oscillatory Solutions

Consider the case where the imposed pressure is of the form

$$\bar{p} = \rho g f(h) \cos \omega t \tag{29}$$

where $f(h)$ is positive, has the dimension of a length, and is a function of the depth h and also of horizontal coordinates. Searching for oscillatory solutions, Eqs. (27) and (29) give

$$\Delta = \frac{\alpha}{3} a \cos \omega t \tag{30}$$

where

$$\alpha = -\frac{g\rho}{\gamma p_1} \frac{f(h)}{1 - \omega^2/\Omega^2} \tag{31}$$

Substitution in Eq. (5) gives

$$\frac{\partial}{\partial t}\left[(a + 3\Delta)\dot{\xi}\right] = -ag\left[2 + \alpha f'(h) + \text{oscillatory terms}\right] \tag{32}$$

where $f'(h) = (\partial f/\partial h)$. The right-hand side is purely oscillatory only if $2 + \alpha f' = 0$, or

$$\frac{f' f}{p_1} = \frac{2\gamma}{\rho g}(1 - \omega^2/\Omega^2) \tag{33}$$

which can be solved for the depths h of oscillatory solutions.

Stability of Oscillatory Solutions

To decide on the stability of such solutions, consider the situation at a depth $h + dh$ which differs slightly from a root of Eq. (33). In such a case

$$\frac{\partial}{\partial t}\left[(a + 3\Delta)\dot{\xi}\right] = \left[-ag\frac{\partial}{\partial h}(2 + \alpha f')\right] dh + \text{oscillatory terms} \tag{34}$$

If the coefficient of dh in this expression is positive, the bubble will migrate away from the oscillation level h and the solution is unstable; but if the coefficient is negative, the disturbed bubble will return to the original state and the solution is stable. The condition for stability is, after substitution,

$$\frac{\partial}{\partial h}\left(\frac{ff'}{p_1}\right) < 0 \qquad (35)$$

Noting $p_1 = p_o + g\rho h$, the condition may be written

$$ff'' < -f'^2 + \frac{\rho g}{p_1} f' f \qquad (36)$$

In the case of a rigid vessel $f(h) = Nh$, and stability would require

$$0 < -1 + \frac{\rho g h}{p_1}$$

which can never be satisfied because $p_1 > \rho g h$.

One can investigate, similarly, stability against disturbance in the horizontal plane, say in a direction r. The equation for the migration follows from Eq. (28) by setting $g = 0$ and replacing h by r. Oscillatory solution can only occur if

$$\frac{\partial f}{\partial r} = 0 \qquad (37)$$

The condition for their stability becomes

$$f\frac{\partial^2 f}{\partial r^2} < 0 \qquad (38)$$

f being positive, these two equations will be satisfied at points on any level h where f is a maximum. As the imposed vibration pressure is proportional to f, it follows that the bubble will oscillate in a stable motion if Eq. (36) is satisfied and if the pressure amplitude is a maximum compared to other points on the same level h.

It must also be stressed that in locations where the oscillations are stable, bubbles above the level h will sink, and those below will rise. This is the exact opposite of what was found in the case of a rigid vessel, where the only possible oscillatory motion was unstable.

Application to a Cylindrical Tank

The pressure field $\overline{p}$ for the case of an elastic cylindrical tank, figure 4, has been studied in reference 5. In the range of frequencies of present interest the pressure distribution is very well approximated by an expression*

$$f(h) = \overline{C}\,\frac{R}{\mu}\sin\left(\mu\frac{h}{R}\right) I_o\left(\mu\frac{r}{R}\right) \qquad (39)$$

where $\overline{C}$ and μ are constants, r is the radial coordinate, R the tank radius, and I_o denotes the modified Bessel Function. The constant $\overline{C}$ defines the magnitude of the pressure, while μ is defined in reference 5; it depends on the properties of the tank** and on the forcing frequency ω.

The depth h of oscillatory solutions can now be determined from Eq. (33), which becomes

$$\frac{\sin\left(2\mu\frac{h}{R}\right)}{\frac{2\mu p_o}{R\rho g} + \frac{2\mu h}{R}} = \frac{2\gamma}{c^2}\left(1 - \frac{\omega^2}{\Omega^2}\right) \qquad (40)$$

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* According to reference 5, this expression is a good approximation except in a region near the bottom of the tank where $h > L - R/2$.

** For the special case of a rigid tank one finds $\mu = 0$, and Eq. (39) becomes in the limit $f(h) = \overline{C}h$. Substitution in Eq. (29) shows that in this case $\overline{C} = N$, where Ng is the peak acceleration applied to the rigid tank.

where

$$C = \overline{C} I_o \left(\mu \frac{r}{R}\right) \tag{41}$$

Equation (40) is graphically represented in figure 5 in a typical manner. If the constant C, representing the applied acceleration, is large enough there will be two roots $2\mu \frac{h}{R} < \pi$ (or one double root), and possibly further roots above 2π. To determine the stability one could use Eq. (36). However, Eq. (35) from which (36) was derived is identical with the statement that the derivative of the left side of Eq. (40) be negative; as the slope of this curve in figure 5 for the first root is necessarily positive, it will be unstable, while the next larger root where the slope is negative is stable. The third root, if any, will again be unstable, etc. The roots occur in pairs of one unstable and one stable root; if a double root occurs it is easily seen that it is unstable.

The question of the location of a point of stable oscillation in the horizontal plane remains to be considered. According to Eq. (38) it is necessary that the pressure function f be a maximum considered as a function of r; the modified Bessel function $I_o(\mu\, r/R)$ in Eq. (39) has a minimum for $r = 0$, but no maximum at all, indicating that there are no locations of stable oscillations in the interior of the tank. The reasoning which leads to the stability criterion (35) indicates that bubbles will move away from points of unstable oscillations, applied to the present case, they will therefore move away from the axis of the tank, towards the walls. In the vicinity of the walls the present theory ceases to be valid, and to obtain a complete picture the effect of surfaces must be included, at least in a qualitative manner.

Effect of Surfaces on the Motion of Bubbles

The effect of a neighboring rigid surface on the motion of an oscillating bubble has been determined by the use of the image principle in reference 1. The major effect found is an attraction of the bubble by the field of the image.

In first approximation the bubble obtains an additional velocity $\dot{\xi}_M$ directed towards the rigid surface,

$$\dot{\xi}_M = -\frac{3}{4H^2} a^{-2} \frac{d\bar{a}}{dt} + \frac{3}{2H^2 a^{-3}} \int_0^t a^{-4} \left(\frac{d\bar{a}}{dt}\right)^2 dt \qquad (42)$$

where H is the distance of the center of the bubble from the surface, and $\bar{a} = a + \Delta$ is the instantaneous radius of the bubble. If the radius $\bar{a}$ varies harmonically the first term is just oscillatory and therefore not important, while the second term is the continuing migration towards the surface.

While Eq. (42) was derived for a rigid surface, it should apply approximately also to the case of the walls of an elastic vessel provided the bubble is sufficiently small. This conclusion will hold provided that the mass of the wall area affected by the pressure field* of the bubble is appreciably larger than the virtual mass of the bubble.

It is therefore concluded that small bubbles will be subject to an additional motion towards the walls of the vessel which is superimposed on the motion previously determined. This additional motion is proportional to $1/H^2$ and is therefore appreciable only if the distance H is of the order of the radius a.**

Discussion of the Result for an Elastic Cylindrical Tank

Equation (40) can be used to determine the regions of different bubble behavior in an elastic cylindrical tank. For a given frequency and externally-imposed acceleration of the tank, the constants $\overline{C}$ and μ in Eq. (39) are computed according to reference 5. Solutions h of Eq. (40), if any, will also be functions of the location r; these solutions will define surfaces separating regions of different bubble behavior.

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* The order of magnitude of this area is H^2.

** A more detailed treatment will be included in a later report.

Figure 5, representing Eq. (40), in a typical manner indicates that for small values of C, that is for small accelerations, no solution h exists. Noting that $C = \overline{C} I_0(\mu r/R)$ is also a function of the radius, it is seen that the value C is a minimum for $r = 0$ and increases towards the walls, $r \rightarrow R$. Therefore, if the acceleration is gradually increased, a range will be reached where Eq. (40), while having no root for $r = 0$, does have solutions for larger values of r. The resulting situation is shown in figure 6a. For a somewhat larger acceleration, Eq. (40) will have roots for any value of r, and two separate surfaces as shown in figure 6b are obtained.

The behavior in the regions separated by these surfaces can best be described by an indication of the direction of the vertical component of the motion in the various regions. As previously indicated, there will also be horizontal motions--from the centerline towards the walls--in all regions. As the upper surface A, figure 6b, represents locations of "unstable" oscillations, bubbles will move away from it; the reverse applies to the lower surface B. It follows that bubbles in the shaded volume between the two surfaces will move downward and outward, while the motion elsewhere will be upward and outward. In the vicinity of the walls of the vessel the outward velocity will be increased by the additional local motion towards the wall. Bubbles below the upper surface A can be expected to find their way towards C, where the lower surface B meets the wall of the tank.

In the transition case shown in figure 6a, bubbles in the shaded region will again move downwards towards C, while those in the lower part of the tank may go towards C or vent, depending on their radial location.

One further case which should be mentioned is shown in figure 6c. It is possible that the lower surface B, defined by the second root of Eq. (40), lies outside the tank. In this case bubbles will simply move down towards the bottom and remain there. They will not move towards the wall because the pressure distribution (39), from which the outward motion elsewhere is derived, does not apply close to the bottom. The situation shown in figure 6c is typical for a rather rigid tank where the parameter μ is quite small; it applies for the pilot tests described in the Introduction and in Appendix A.

In very flexible tanks more than two separating surfaces may occur, as shown in figure 7. In such cases two or more circles C, C' are potential locations for the collection of bubbles.

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APPENDIX A
OBSERVATIONS ON A VIBRATED TRANSPARENT TANK

To study the phenomena tentatively, a cylindrical transparent plastic container partially filled with water, figure 1, was vibrated vertically by means of a vibration table through a range of frequencies and accelerations. A typical result was as follows: Setting the machine at 80 cycles/sec the acceleration was increased gradually. At first only ripples of various patterns occurred, which were followed by droplets being thrown upwards. Beginning at about 11 g acceleration--fully developed at about 14 g--an antisymmetrical surface motion of roughly 160 cycles/min appeared, figure 1A; to the eye it seemed to be a rotary motion, but it was judged to be a combination of two antisymmetric motions at right angles to each other having different phases. The amplitude stabilized at about $\pm$ 25 degrees, the observed frequency being slightly smaller than the computed one (190/min) for antisymmetrical <u>small</u> vibrations in a rigid vessel.

When this antisymmetrical motion was nearly fully developed, at 13 g, some gas bubbles of about 1/10-inch diameter detached themselves from the surface and travelled rather slowly downwards; after reaching a level of 2 - 3 inches below the surface most bubbles went upwards again; occasionally one clung to the side or was observed making circular motions at the bottom. As the acceleration was increased some bubbles seemed suspended 3 - 5 inches below the surface; these bubbles executed two simultaneous vertical vibratory motions: one of the forcing frequency which made them appear oblong in the vertical direction (this was resolved with a stroboscope); the other motion had about the frequency of the antisymmetrical surface mode and about 1/8-inch amplitude. When the acceleration reached 24 g a stream of bubbles of nearly equal size descended to the bottom; if the acceleration was maintained for one-half to one minute a resonance situation set in, forcing stoppage of the test; if the acceleration was reduced after a few seconds a cluster of bubbles could be maintained at the bottom, and sometimes a few inches above the bottom clinging to the wall. If the vibration was stopped at any time all existing bubbles rose rapidly to the surface.

Similar phenomena, occurring in the same sequence but at different accelerations, were observed in the range from 35 to 150 cycles/second. The matter of principal interest here is the fact that bubbles formed in the surface region will appear and remain at lower levels, contrary to gravitational forces. The tests show clearly, also, that in a strongly vibrated vessel bubbles are attracted by the sides and bottom of the vessel, and also by each other. Bubbles on the bottom, for example, were regularly seen moving in groups, close to each other, without merging. Clusters of bubbles formed in the final stages remained together also, clinging to either the wall or the bottom of the tank.

Two points concerning the bubbles might be emphasized without attempting any interpretation at this time. First, it appeared that the size of the bubbles observable at any time was identical, as far as the eye could tell, particularly in the final stage when a stream of bubbles descended from the surface. Secondly, in the tests witnessed by the writer, bubbles were not observed, except at the surface, until after the antisymmetrical surface motion was quite noticeable.

Apart from the behavior of bubbles and bubble clusters, the writer was struck by the pronounced antisymmetrical motion of the surface at and above critical values of the applied acceleration. One does not expect to excite antisymmetric modes by the axial, that is symmetric, motion; yet the observations do not permit dismissal of the existence of these antisymmetrical modes as accidental. The accelerations at which these motions started, and became fully developed, respectively, are given in the following Table A for various frequencies.

TABLE A

Cycles/second	30	40	80	100	150
Acceleration in g	4 - 5	? - 7	11 - 14	15 - 20	20 - 24

It is planned to study the question of the origin of these sloshing motions because the matter may have consequences quite removed from the bubble problem.

APPENDIX B

EVALUATION OF THE LAST INTEGRAL IN EQUATION (1)*

Let $dV = d\xi\, dA$ where ξ is a vertical coordinate, figure 3A, and dA is an element of area in the horizontal plane

$$\iiint (u_\Delta + u_z)\, dV = \int \left[\iint (u_\Delta + u_z)\, dA \right] d\xi \qquad \text{(a)}$$

The integral in the bracket is the downward flux F through a plane on the level ξ due to the combined motions $\dot{\Delta}$ and $\dot{z}$, and the integral in Eq. (a) equals $\int F\, d\xi$. If the plane is below the bubble the flux F must be zero because the fluid is incompressible. If the plane is above the bubble, or cuts the instantaneous position of the bubble, the flux $F(\xi)$ is the negative rate of change, $(-\partial B/\partial t)$, of the partial volume $B(\xi)$ of the bubble below the plane, figure 3A. Therefore,

$$\iiint (u_\Delta + u_z)\, dV = -\int_0^{z+a+\Delta} \frac{\partial B(\xi)}{\partial t}\, d\xi = -\frac{\partial}{\partial t}\int_0^{z+a+\Delta} B(\xi)\, d\xi$$

Integrating by parts and noting that the boundary terms vanish because $B(z + a + \Delta) = 0$,

$$\iiint (u_\Delta + u_z)\, dV = \frac{\partial}{\partial t}\int_0^{z+a+\Delta} \xi \frac{\partial B(\xi)}{\partial \xi}\, d\xi \cong -\frac{\partial}{\partial t} \frac{4\pi}{3} (a + \Delta)^3 z \qquad \text{(b)}$$

The value of the last integral was obtained by noting that it represents the negative first moment of the total bubble volume with respect to the surface.

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* One might be tempted to compute this integral simply by substituting the known values of u_Δ and u_z for an infinite unbounded fluid. This would lead to an incorrect result as the effect of the boundary, even if very far, contributes substantially to the value of this integral. The method of evaluation used here includes this effect.

APPENDIX C
EFFECT OF VISCOSITY

As the buoyancy forces are proportional to a^3, while viscosity forces change as the surface, that is a^2, viscosity will enter the problem, and even control for sufficiently small bubbles. The viscous force resisting the displacement of a bubble of radius $a + \Delta$ can be expected to be of the form $C(a + \Delta)^2 z$ where C is a constant. This force can be included in the analysis modifying Eq. (14):

$$\frac{\partial}{\partial t}(a + \Delta)^3 \dot{\xi} + C(a + \Delta)^2 \dot{\xi} = 2(a + \Delta)^3 (\ddot{x} - g) \qquad (a)$$

while Eq. (15) remains unchanged. The problem can be treated as before, and oscillatory solutions exist in certain cases. The character of the solution changes only with regard to the phase of ξ with respect to x.

To see the nature of the modification consider the extreme case of very small bubbles such that the first term in Eq. (a), representing inertia effects, can be dropped:

$$\dot{\xi} = \frac{2}{C}(a + \Delta)(\ddot{x} - g) \qquad (b)$$

Substituting the solution (16) for Δ one finds easily that oscillatory solutions exist, under the same assumptions as before, but provided that

$$\alpha N = 6 \qquad (c)$$

instead of the previous condition $N = 2$. It is interesting to note that the limiting condition (c) does not depend on the value of C. When the critical depth h is computed about 3 times larger values are found than before.

Caution is required, however, when using Eq. (a). By applying it to the steady rise of a bubble in a constant gravity field, it can be shown that this equation is valid only for quite small bubbles. Equation (a) leads to a linear relation between the radius and the terminal velocity which agrees with experiments for air bubbles in water only for radii of less than 0.1 cm; for larger bubbles the terminal velocity increases only slowly because complicated phenomena, like spiraling, occur (Eq. 6).

In view of the above, just a rough clue as to whether or not viscosity will modify the results of the analysis of the main body of this paper can be obtained by comparing the velocity $\dot{\xi}$, Eq. (19), and available information on the terminal velocity of rising bubbles. For bubbles between one-tenth to one inch in diameter, the terminal velocity of air bubbles in water is 8 to 10 inches per second. Unless the velocity $\dot{\xi}$ is smaller than this value, the actual depth h can be expected to be larger than that computed from Equations (24) and (25).

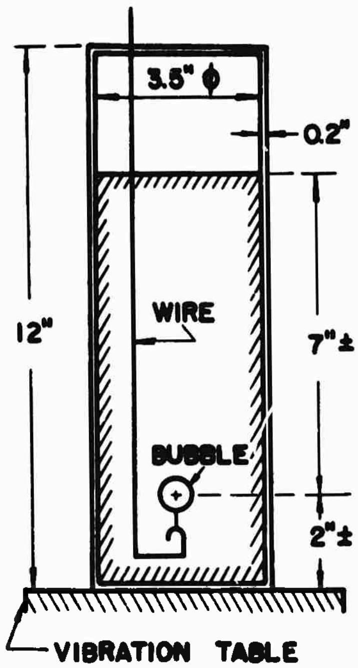

FIG. 1

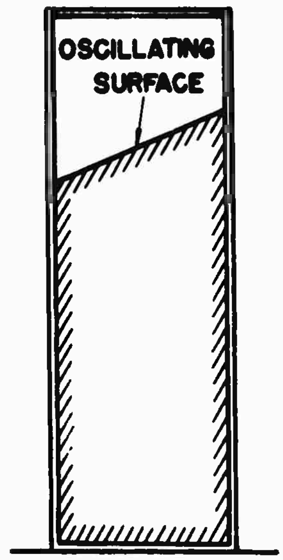

FIG. 1A

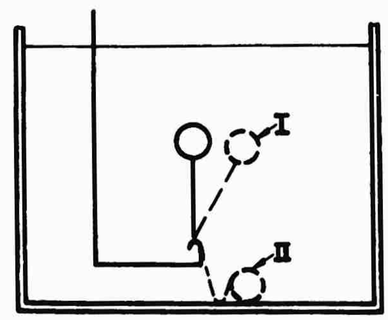

FIG. 2

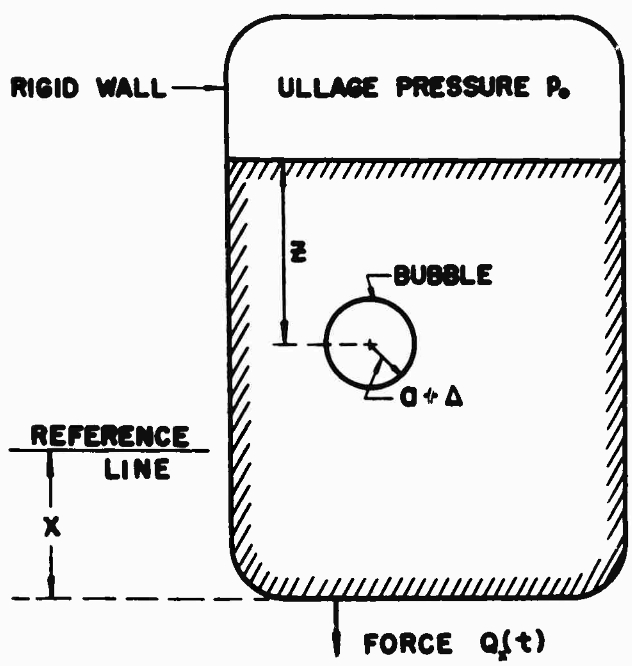

FIG. 3

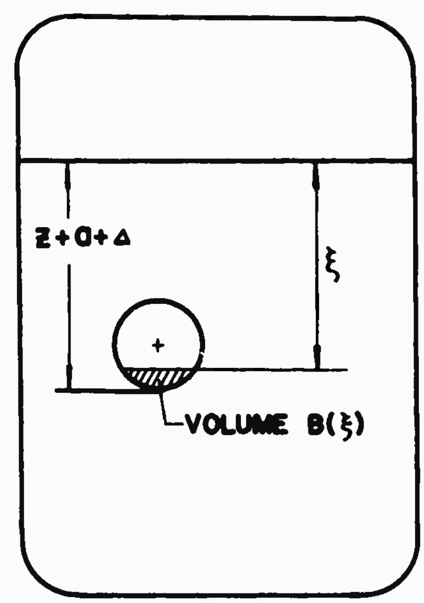

FIG. 3A

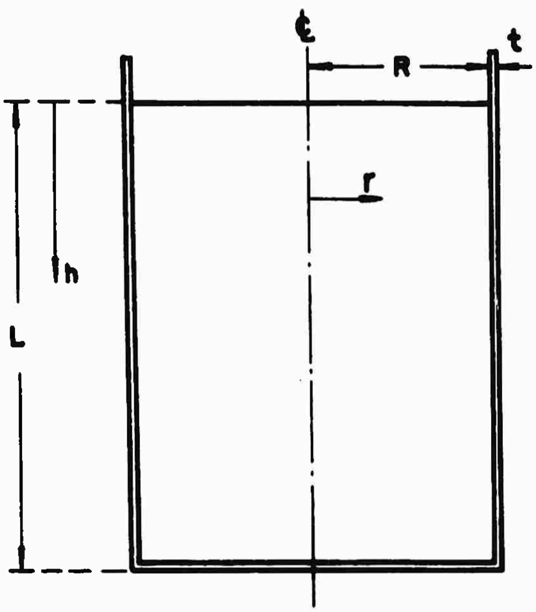

FIG. 4

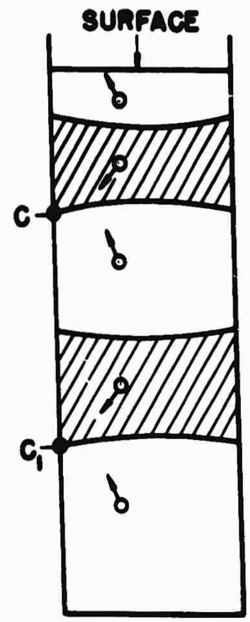

FIG. 7

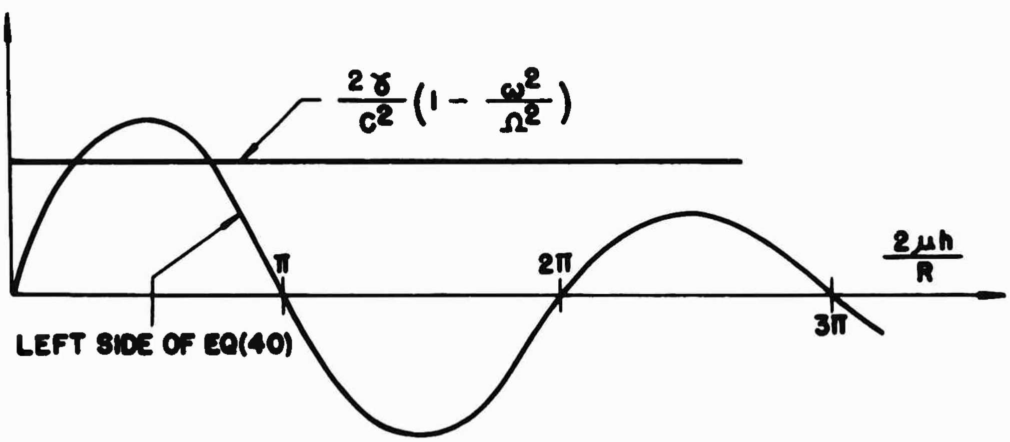

FIG. 5

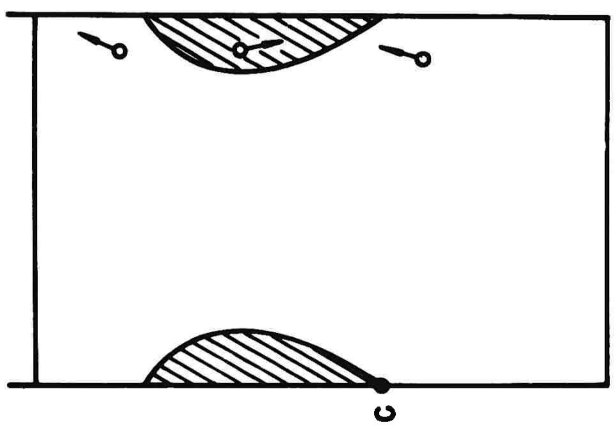

FIG.6a

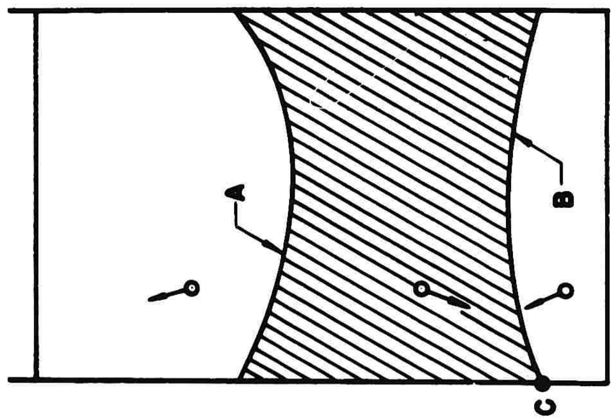

FIG.6b

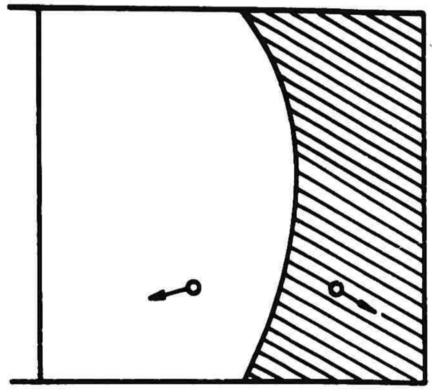

FIG.6c

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